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# Algebraic shape invariant potentials in two steps

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**Abstract**

The simplified algebraic structure of the shape invariance condition in two steps is developed by imposing an extra relation on the two superpotentials of the corresponding two-step potentials. This simplified version of potential algebra is found to be similar to that of the shape invariance condition in one step. The solvable potentials of shape invariance in two steps, with a translation change of parameters $a_1 = a_0 + \delta$, are shown to possess such a simplified version of potential algebra. The condition under which the potential algebra of shape invariance in more than two steps becomes simplified is also discussed.

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## 1. Introduction

The idea of supersymmetry was initially introduced to solve a hierarchy problem in grand unified theories. Supersymmetry (SUSY) is not an exact symmetry of Nature. However, it is difficult to determine whether SUSY is broken non-perturbatively in supersymmetric quantum field theories. Supersymmetric quantum mechanics (SUSYQM) therefore becomes a testing ground to understand non-perturbatively SUSY breaking [1, 2]. For a review of SUSYQM, please refer to [3–5] and references therein.

In SUSYQM, one constructs a SUSY partner Hamiltonian, starting from a given one-dimensional Hamiltonian by the method of factorization [6, 7]. The process can even be used successively to generate an entire hierarchy of isospectral Hamiltonians. Let us be more specific. Two potentials $V_-(x, a_0)$ and $V_+(x, a_0)$ are said to be SUSY partner potentials, if they are related to the superpotential $W(x, a_0)$ by

$$V_\pm(x, a_0) = W^2(x, a_0) \pm W'(x, a_0), \tag{1}$$

where $W'(x, a_0) \equiv \frac{\mathrm{d}}{\mathrm{d}x} W(x, a_0)$ and $a_0$ is a set of parameters. In units of $\hbar = 2m = 1$, the corresponding SUSY partner Hamiltonians $H_-(a_0)$ and $H_+(a_0)$ have the following factorizable forms:

$$H_-(a_0) = A^\dagger(a_0)A(a_0), \qquad H_+(a_0) = A(a_0)A^\dagger(a_0), \tag{2}$$

where the operators $A(a_0)$ and $A^\dagger(a_0)$ are defined as

$$A(a_0) = \frac{\mathrm{d}}{\mathrm{d}x} + W(x, a_0), \qquad A^\dagger(a_0) = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x, a_0), \quad \text{respectively.} \tag{3}$$

Hereinafter, to simplify the notation we have suppressed the $x$-dependence in the Hamiltonians $H_\pm(x, a_0)$ and in the operators $A(x, a_0)$ and $A^\dagger(x, a_0)$. In the case of unbroken SUSY, the unnormalized wave function

$$\psi_0^{(-)}(x, a_0) \propto \exp\left[-\int^x W(y, a_0)\,\mathrm{d}y\right] \tag{4}$$

is the nodeless zero-energy ground-state eigenfunction for $H_-(a_0)$, as can be checked by the equation $A(a_0)\psi_0^{(-)}(x, a_0) = 0$. Except for this zero-energy eigenstate, the SUSY partner Hamiltonians $H_\pm(a_0)$ are found to be isospectral. Explicitly, the eigenstates of $H_-(a_0)$ and $H_+(a_0)$ are related to each other by $\psi_{n-1}^{(+)}(x, a_0) \propto A(a_0)\psi_n^{(-)}(x, a_0)$ and $\psi_{n+1}^{(-)}(x, a_0) \propto A^\dagger(a_0)\psi_n^{(+)}(x, a_0)$, and the eigenvalues are related by $E_{n-1}^{(+)} = E_n^{(-)}$ and $E_0^{(-)} = 0$, where $n = 1, 2, 3, \ldots$. In this way, if we know all the eigenstates of $H_-(a_0)$ we can determine all the eigenstates of $H_+(a_0)$, and vice versa, except for the zero-energy ground-state eigenfunction, equation (4).

SUSYQM has been shown to provide a key ingredient to explore exactly solvable potentials for Schrödinger equations in nonrelativistic quantum mechanics. In this respect, the concept of the shape invariance condition [8] in the formalism of SUSYQM becomes critical, because it leads immediately to an integrability condition to the problem. What we mean by shape invariance is that the pair of partner potentials $V_\pm(x, a_0)$ defined in equation (1) are similar in shape but differ only up to a change of parameters and additive constants. Mathematically, the condition reads[1]

$$V_+(x, a_0) = V_-(x, a_1) + R(a_0), \tag{5}$$

where $a_1 = f(a_0)$ is a function of $a_0$ and the remainder $R(a_0)$ is independent of $x$. By equation (5), the entire spectrum of eigenenergies for the Hamiltonian $H_-(a_0)$ (2) are obtained algebraically as $E_0^{(-)} = 0$ and $E_n^{(-)} = \sum_{k=0}^{n-1} R(a_k)$, where $n = 1, 2, 3, \ldots$ [8, 9]. Moreover, the corresponding $n$th unnormalized eigenfunction is

$$\psi_n^{(-)}(x, a_0) \propto A^\dagger(a_0)A^\dagger(a_1)\cdots A^\dagger(a_{n-1})\psi_0^{(-)}(x, a_n),$$

where the zero-energy eigenfunction $\psi_0^{(-)}(x, a_0)$ is given in equation (4).

Many interesting classes of solvable shape invariant potentials in one step that retain SUSY have been constructed and discussed [10–14], including all the analytically solvable potentials known in the context of nonrelativistic quantum mechanics. We emphasize here that the shape invariance condition is not the most general integrability condition, since some exactly solvable potentials are shown not to be shape invariant [10]. In addition, the shape invariance condition is also studied in the so-called fractional SUSYQM of order $k$ ($k = 3, 4, \ldots$), which generalizes the $Z_2$-grading of the relevant Hilbert space to a $Z_k$-grading [15].

To obtain more solvable shape invariant potentials, the concept of shape invariance can be extended to two and even multi-steps [13]. Based on this method, various shape invariant potentials in two steps have been reported [13, 16, 17]. Some of the obtained solvable potentials are shown to be inherently singular with Dirac delta-function singularities.

The extension to shape invariance in multi-steps is rather straightforward. So, let us consider the shape invariance condition in two steps with unbroken SUSY. We are given

[1] Since the partner potentials $V_-(x, a_1)$ and $V_+(x, a_0)$ are related to each other by one relation, equation (5) is thus called the one-step shape invariance condition.

two superpotentials $W(x, a_0)$ and $\tilde{W}(x, a_0)$ such that the respective potentials $V_\pm(x, a_0)$ and $\tilde{V}_\pm(x, a_0)$ defined in equation (1) obey the following relations:

$$V_+(x, a_0) = \tilde{V}_-(x, a_0) + R(a_0), \tag{6}$$

$$\tilde{V}_+(x, a_0) = V_-(x, a_1) + \tilde{R}(a_0). \tag{7}$$

By shape invariance condition in two steps, we mean that the SUSY partner potentials $V_-(x, a_1)$ and $V_+(x, a_0)$ now are related to each other by the above two relations.

From equations (6) and (7), the energy eigenvalues and eigenfunctions for the initial potential $V_-(x, a_0)$ can be algebraically constructed. The eigenvalues are found to be

$$E_{2n}^{(-)} = \sum_{k=0}^{n-1}[R(a_k) + \tilde{R}(a_k)], \qquad E_{2n+1}^{(-)} = \sum_{k=0}^{n-1}[R(a_k) + \tilde{R}(a_k)] + R(a_n), \tag{8}$$

where $E_0^{(-)} = 0$ and $n = 0, 1, 2, \ldots$. Further, the corresponding eigenfunctions are

$$\psi_{2n}^{(-)}(x, a_0) \propto [A^\dagger(a_0)\tilde{A}^\dagger(a_0)]\cdots[A^\dagger(a_{n-1})\tilde{A}^\dagger(a_{n-1})]\psi_0^{(-)}(x, a_n),$$
$$\psi_{2n+1}^{(-)}(x, a_0) \propto [A^\dagger(a_0)\tilde{A}^\dagger(a_0)]\cdots[A^\dagger(a_{n-1})\tilde{A}^\dagger(a_{n-1})]A^\dagger(a_n)\tilde{\psi}_0^{(-)}(x, a_n),$$

where the functions $\psi_0^{(-)}(x, a_0)$ and $\tilde{\psi}_0^{(-)}(x, a_0)$, expressible in terms of the corresponding superpotentials by using equation (4), denote the zero-energy ground-state eigenfunctions for the potentials $V_-(x, a_0)$ and $\tilde{V}_-(x, a_0)$, respectively.

The purpose of the present paper is to explore possible algebraic properties of shape invariance condition in two steps. The general solution to this problem remains unsolved. However, as we shall show that if extra relation is introduced between the two superpotentials of the corresponding potentials in two steps, the underlying potential algebra will be simplified and consequently be solved. This simplified version of potential algebra is found to be similar to that of shape invariance in one step. In essence, the introduction of extra relations effectively reduces the shape invariance condition from two steps to one step. The solvable shape invariant potentials in two steps, with a translation change of parameters ($a_1 = a_0 + \delta$), discussed in [16, 17] are shown to possess such simplified version of potential algebra. The paper is organized as follows. In section 2, by imposing extra relations, we establish the simplified version of the potential algebra of the shape invariance condition in two steps. In section 3, we explicitly work out two classes of solvable shape invariant potentials in two steps, which are shown to possess such simplified potential algebra. Finally, we conclude the paper in section 4, along with the discussion on the simplified potential algebra of the shape invariance condition in multi-steps.

## 2. Algebraic properties of shape invariance in two steps

It is known that the shape invariance condition in one step (5) possesses what is generally referred to as a potential algebra [18, 19]. That is, the shape invariant potentials have an underlying algebraic structure, thus can also be studied by group theoretical methods [20, 21]. In the potential algebraic approach, the representation of the associated algebra is constructed based on three angular-momentum-like generators. Moreover, the shape invariant Hamiltonian of a system is expressed by a Casimir operator of the associated algebra, and the energy eigenvalues are determined by purely algebraic means. So far, the potential algebras are mainly exploited in the shape invariance condition in one step. Not many efforts are focused on the shape invariant potentials in two and even multi-steps. Hence, we shall explore in this section the possible connection between the shape invariance condition in two steps and its underlying potential algebra in a concrete and consistent way. We adopt the notations of [16] in the following.

To begin with, let us consider in equations (6) and (7) the substitution of parameters $a_0 \to \boldsymbol{\alpha}(N)$ and for the general case the substitution $a_m \to \boldsymbol{\alpha}(N-m)$, where $N$ is an arbitrary integer and $m = 0, 1, 2, \ldots$. The function $\boldsymbol{\alpha}(N) = \{\alpha_1(N), \alpha_2(N)\}$ has two components, and is determined by requiring that the change $\boldsymbol{\alpha}(N) \to \boldsymbol{\alpha}(N-1)$ corresponds to the change of parameters $a_0 \to a_1$. The detailed forms of $\boldsymbol{\alpha}(N)$ will be given in section 3. In terms of the superpotentials $W(x, \boldsymbol{\alpha}(N))$ and $\tilde{W}(x, \boldsymbol{\alpha}(N))$, equations (6) and (7) now become

$$W^2(x, \boldsymbol{\alpha}(N)) + W'(x, \boldsymbol{\alpha}(N)) = \tilde{W}^2(x, \boldsymbol{\alpha}(N)) - \tilde{W}'(x, \boldsymbol{\alpha}(N)) + R(\boldsymbol{\alpha}(N)), \tag{9}$$

$$\tilde{W}^2(x, \boldsymbol{\alpha}(N)) + \tilde{W}'(x, \boldsymbol{\alpha}(N)) = W^2(x, \boldsymbol{\alpha}(N-1)) - W'(x, \boldsymbol{\alpha}(N-1)) + \tilde{R}(\boldsymbol{\alpha}(N)). \tag{10}$$

Both superpotentials $W(x, \boldsymbol{\alpha}(N))$ and $\tilde{W}(x, \boldsymbol{\alpha}(N))$ are quite arbitrary at this stage. Without further clues, obviously it is too difficult to determine the underlying potential algebra described by these two equations.

To solve this problem, we restrict ourselves to a particular subset of the shape invariance condition in two steps, instead. The subset is defined by imposing extra relations on the two superpotentials and on the two remainders in equations (9) and (10). This will result in a simplified version of the potential algebra. Explicitly, the particular relations are based on the identification:

$$\tilde{W}(x, \boldsymbol{\alpha}(N)) \equiv W\left(x, \boldsymbol{\alpha}\left(N - \tfrac{1}{2}\right)\right), \qquad \tilde{R}(\boldsymbol{\alpha}(N)) \equiv R\left(\boldsymbol{\alpha}\left(N - \tfrac{1}{2}\right)\right). \tag{11}$$

Then, with the help of equation (11), the two seemly unrelated equations (9) and (10) can be cast into a single and compact equation as

$$W^2(x, \boldsymbol{\alpha}(N)) + W'(x, \boldsymbol{\alpha}(N)) = W^2\left(x, \boldsymbol{\alpha}\left(N - \tfrac{1}{2}\right)\right) - W'\left(x, \boldsymbol{\alpha}\left(N - \tfrac{1}{2}\right)\right) + R(\boldsymbol{\alpha}(N)). \tag{12}$$

We note that equation (12) now is identical to equation (9), and equation (10) can be obtained by replacing $N$ by $N - \frac{1}{2}$ in that equation.

Equation (12) actually represents a constraint equation for the superpotential $W(x, \boldsymbol{\alpha}(N))$, when the parameter $N$ is changed by one-half as we go from one superpotential to another. In the viewpoint of quantum mechanics, this change of parameter, i.e., $N \to N - \frac{1}{2}$, can be formally accomplished by the action of the raising and lowering operators of angular momentum [16, 20]. With this in mind, we first define an operator $J_3$, that is analogous to the $z$-component angular momentum operator, by

$$J_3 \equiv \mathrm{i}\frac{\partial}{\partial \phi}, \tag{13}$$

and designate the parameter $N$ as the eigenvalue of the operator $J_3$ acting on the eigenstate $|N\rangle$, that is, $J_3\,|N\rangle = N\,|N\rangle$. Now, it is natural to think that the superpotential function lives in a two-dimensional space spanned by the coordinates $x$ and $\phi$, and the eigenvalue of $J_3$ is assigned to the labeling parameter. In this way, equation (12) can be immediately generated, when we project this operator equation

$$W^2(x, \boldsymbol{\alpha}(J_3)) + W'(x, \boldsymbol{\alpha}(J_3)) = W^2\left(x, \boldsymbol{\alpha}\left(J_3 - \tfrac{1}{2}\right)\right) - W'\left(x, \boldsymbol{\alpha}\left(J_3 - \tfrac{1}{2}\right)\right) + R(\boldsymbol{\alpha}(J_3)), \tag{14}$$

into the eigenstate $|N\rangle$.

Next, we construct the ladder-type operators of angular momentum. The lowering operator $J_-$ and the raising operator $J_+ = (J_-)^\dagger$ are built using the superpotential $W(x, \boldsymbol{\alpha}(J_3))$ as

$$J_- = \left[\frac{\partial}{\partial x} + W(x, \boldsymbol{\alpha}(J_3))\right]\mathrm{e}^{\mathrm{i}\phi/2}, \qquad J_+ = \mathrm{e}^{-\mathrm{i}\phi/2}\left[-\frac{\partial}{\partial x} + W(x, \boldsymbol{\alpha}(J_3))\right]. \tag{15}$$

The operator $J_-(J_+)$ basically has the same form as the operator $A(a_0)(A^\dagger(a_0))$ described in equation (3), respectively.

Explicit computations then show that the simplified potential algebra of shape invariance in two steps based on the set of generators $\{J_+, J_-, J_3\}$ is indeed closed and generically nonlinear. Their commutation relations are found to be

$$[J_3, J_\pm] = \pm\tfrac{1}{2}J_\pm, \qquad [J_+, J_-] = -R(\boldsymbol{\alpha}(J_3)), \tag{16}$$

where we have used equation (14) and the operators $J_\pm$ change the eigenvalues of $J_3$ by one-half, as desired.

The potential algebra (16) based on the set of generators $\{J_+, J_-, J_3\}$ looks very familiar. Surely, it defines the potential algebra of the shape invariance condition in one step [18–21]. If the remainder $R(\boldsymbol{\alpha}(J_3))$ is linear in $J_3$, the resultant potential algebra will reduce to that of so (3) or so (2,1). However, if $R(\boldsymbol{\alpha}(J_3))$ is nonlinear in $J_3$, the underlying potential algebra is born nonlinear. We therefore ascertain that the identification (11) indeed simplifies the associated potential algebra defined by equations (9) and (10), and reduces the shape invariance condition effectively from two steps to one step.

The representation of the simplified version of potential algebra (16) can be obtained by considering the simultaneous eigenstates, denoted by $|s\rangle$ with $s = N - \frac{m}{2}$, of the mutually commuting operators $H_-(\boldsymbol{\alpha}(J_3 - \frac{1}{2})) = J_+J_-$ and $J_3$. The respective eigenvalue equations are denoted by

$$H_-\left(\boldsymbol{\alpha}\left(J_3 - \tfrac{1}{2}\right)\right)|s\rangle = \varepsilon^2(\boldsymbol{\alpha}(s))\,|s\rangle\,, \qquad J_3\,|s\rangle = s|s\rangle. \tag{17}$$

Both equations together imply the action of ladder-type operators $J_-$ and $J_+$ on the eigenstate $|s\rangle$ as

$$J_-\,|s\rangle = \varepsilon(\boldsymbol{\alpha}(s))\left|s - \tfrac{1}{2}\right\rangle, \qquad J_+\,|s\rangle = \varepsilon\left(\boldsymbol{\alpha}\left(s + \tfrac{1}{2}\right)\right)\left|s + \tfrac{1}{2}\right\rangle, \tag{18}$$

where, without loss of generality, the coefficient $\varepsilon(\boldsymbol{\alpha}(s))$ is chosen to be real. If the spectrum of the Hamiltonian $H_-(\boldsymbol{\alpha}(J_3 - \frac{1}{2}))$ exhibits the lowest-weight eigenstate, that is, $J_-\,|s_0\rangle = 0$, we choose the coefficient $\varepsilon(\boldsymbol{\alpha}(s))$ to satisfy the condition: $\varepsilon(\boldsymbol{\alpha}(s_0)) = 0$ and $s - s_0 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Otherwise, if the spectrum exhibits the highest-weight eigenstate, $J_+\,|s_0\rangle = 0$, then we have to choose $\varepsilon(\boldsymbol{\alpha}(s_0 + \frac{1}{2})) = 0$ and $s - s_0 = 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots$, instead.

Furthermore, the energy eigenvalues of the initial Hamiltonian $H_-(\boldsymbol{\alpha}(N))$ can be expressed purely in terms of the coefficient $\varepsilon(\boldsymbol{\alpha}(s))$. We determine this expression by projecting the right-hand side commutator of equation (16) on the eigenstate $|s\rangle$, and obtain

$$\varepsilon^2(\boldsymbol{\alpha}(s)) - \varepsilon^2\left(\boldsymbol{\alpha}\left(s + \tfrac{1}{2}\right)\right) = -R(\boldsymbol{\alpha}(s)). \tag{19}$$

Then applying this equation recursively, we get from equation (8) that the eigenenergies of the initial Hamiltonian $H_-(\boldsymbol{\alpha}(N))$ (2) are

$$E_n^{(-)} = \sum_{s=N}^{N-\frac{n-1}{2}} R(\boldsymbol{\alpha}(s)) = \varepsilon^2\left(\boldsymbol{\alpha}\left(N + \tfrac{1}{2}\right)\right) - \varepsilon^2\left(\boldsymbol{\alpha}\left(N - \tfrac{n-1}{2}\right)\right), \tag{20}$$

where $n = 0, 1, 2, \ldots$. We recall the identification $\tilde{R}(\boldsymbol{\alpha}(N)) = R\big(\boldsymbol{\alpha}\big(N - \frac{1}{2}\big)\big)$ in equation (11).

## 3. Examples

To illustrate the simplified potential algebra developed in the preceding section, section 2, we explicitly work out two classes of shape invariant potentials in two steps. These shape invariant potentials are known solvable with a translation change of parameters $a_m = a_{m-1} + \delta = a_0 + m\delta$, where $\delta$ is a constant and $m = 0, 1, 2, \ldots$. Nevertheless, they are singular in the sense that discontinuity at some $x$-points is characteristic of the superpotentials and gives rise to Dirac delta-function singularities to the corresponding potentials. The construction of the generators for the associated simplified version of potential algebra, in turns, provides an alternative way of deducing the energy eigenvalues by algebraic formulation.

### *3.1. Solvable singular potentials I*

In the first class, the two superpotentials $W(x, a_m)$ and $\tilde{W}(x, a_m)$ of the corresponding solvable shape invariant potentials in two steps are given by [17]

$$\left.\begin{matrix} W(x, a_m) \\ \tilde{W}(x, a_m) \end{matrix}\right\} = \left(\frac{R_0}{R_1} \mp \frac{\delta}{4} + a_m\right) g(x) \pm \frac{R_1 - \tilde{R}_1}{8g(x)}, \tag{21}$$

where $R_0$, $R_1$ and $\tilde{R}_1$ are constants. The function $g(x)$ is determined by the first-order differential equation

$$g'(x) - \frac{\delta}{2} g^2(x) = \frac{1}{4}(R_1 + \tilde{R}_1). \tag{22}$$

Three SUSY preserving solutions are interesting in this class. As an illustration, we choose $R_1 + \tilde{R}_1 > 0$ to ensure that the zero-energy ground-state wavefunctions $\psi_0^{(-)}(x, a_0)$ and $\tilde{\psi}_0^{(-)}(x, a_0)$ can be made normalizable. The shape invariant potentials in two steps are: (i) for $\delta = 0$, the singular harmonic oscillator potential, (ii) for $\delta > 0$, the singular Pöschl–Teller I potential and (iii) for $\delta < 0$, the singular Pöschl–Teller II potential [16, 22]. In any of these potentials, the superpotentials are continuous with well-defined derivatives everywhere in either $x > 0$ or $x < 0$ regions. At the origin $x = 0$, they are however singular like $x^{-1}$, and have an infinite discontinuity. A regularization that preserves SUSY and shape invariance has to be introduced, thus gives rise to the corresponding regularized potentials a Dirac delta-function singularity at the origin [22, 23].

To present the simplified potential algebra for the superpotentials (21), we adopt the substitution of the parameter $a_m \to \boldsymbol{\alpha}(N - m)$, where $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2\}$ is a two-component parameter and $N$ is an arbitrary integer. Then using the identification (11), we denote $W(x, a_m) \equiv W\big(x, \boldsymbol{\alpha}\big(N - \frac{1}{2}(2m)\big)\big)$ and $\tilde{W}(x, a_m) \equiv W\big(x, \boldsymbol{\alpha}\big(N - \frac{1}{2}(2m+1)\big)\big)$. In this way, the two originally different superpotentials in equation (21) are recast into a single unified superpotential, and are related to each other when the parameter $N$ of the unified superpotential is changed by $\frac{1}{2}$. In brevity, the unified superpotential takes the concise form (for $s \equiv N - \frac{m}{2}$ and $m = 0, 1, 2, \ldots$)

$$W(x, \boldsymbol{\alpha}(s)) = \alpha_1(s) g(x) + \frac{\alpha_2(s)}{g(x)}, \tag{23}$$

where the two components $\alpha_1(s)$ and $\alpha_2(s)$ of $\boldsymbol{\alpha}(s)$ are defined by

$$\alpha_1(s) = \frac{R_0}{R_1} + a_0 + \left(N - s - \frac{1}{4}\right)\delta, \tag{24}$$

$$\alpha_2(s) = \frac{1}{8}(-1)^{2s}(R_1 - \tilde{R}_1). \tag{25}$$

With the definition of the superpotential (23), we explicitly build the angular-momentum-like generators $J_-$, $J_+$ and $J_3$ of the associated simplified potential algebra as

$$J_- = \left[\frac{\partial}{\partial x} + \alpha_1(J_3) g(x) + \frac{\alpha_2(J_3)}{g(x)}\right] \mathrm{e}^{\mathrm{i}\phi/2}, \qquad J_+ = (J_-)^\dagger, \qquad J_3 = \mathrm{i}\frac{\partial}{\partial \phi}. \tag{26}$$

It is straightforward to compute the commutator $[J_+, J_-]$. Taking into account the shape invariance condition (14), we arrive at the following expression for the remainder $R(\boldsymbol{\alpha}(s))$:

$$R(\boldsymbol{\alpha}(s)) = \frac{1}{2}\left(\alpha_1(s) + \frac{\delta}{4}\right)[(R_1 + \tilde{R}_1) + 8\alpha_2(s)]. \tag{27}$$

Then, on solving equations (19) and (27), we establish the general form for the coefficients $\varepsilon^2(\boldsymbol{\alpha}(s))$,

$$\varepsilon^2(\boldsymbol{\alpha}(s)) = C - \frac{1}{2\delta}(R_1 + \tilde{R}_1)\left[\alpha_1\left(s - \frac{1}{2}\right) + \frac{2\delta\alpha_2(s)}{R_1 + \tilde{R}_1}\right]^2, \tag{28}$$

where $C$ is a constant to make $\varepsilon^2(\boldsymbol{\alpha}(s))$ positive definite. Finally, the resultant energy eigenvalues for the initial Hamiltonian $H_-(a_0)$ are obtained in terms of $\varepsilon^2(\boldsymbol{\alpha}(s))$ by using equations (20) and (28).

### *3.2. Solvable singular potentials II*

In the second class, the two solvable superpotentials $W(x, a_m)$ and $\tilde{W}(x, a_m)$ of shape invariance in two steps are described by [17]

$$W(x, a_m) = \left(\frac{R_1 - \tilde{R}_1}{4R_2} + a_m\right)g(x) + \frac{R_1 + \tilde{R}_1}{4R_2}\tilde{g}(x) - \frac{\delta}{2}\frac{g^2(x)}{g(x) + \tilde{g}(x)}, \tag{29}$$

$$\tilde{W}(x, a_m) = \left(\frac{R_1 - \tilde{R}_1}{4R_2} + a_m\right)\tilde{g}(x) + \frac{R_1 + \tilde{R}_1}{4R_2}g(x) + \frac{\delta}{2}\frac{g^2(x)}{g(x) + \tilde{g}(x)}, \tag{30}$$

where $R_1$, $\tilde{R}_1$ and $R_2$ are constants. The function $\tilde{g}(x) = \pm\sqrt{g^2(x) - R_2}$ is determined by $g(x)$ and can have either positive or negative signs. Further, the function $g(x)$ is determined by the first-order differential equation

$$\left(1 + \frac{g(x)}{\tilde{g}(x)}\right)g'(x) - \delta g^2(x) = \frac{1}{2}(R_1 + \tilde{R}_1). \tag{31}$$

Two SUSY preserving solutions are found in the second class. If the conditions $R_2 > 0$, $R_1 + \tilde{R}_1 > 0$ and $R_1 + \tilde{R}_1 + 2\delta R_2 > 0$ are chosen, the shape invariant potentials are: (i) for $\delta = 0$, the singular harmonic oscillator potential, and (ii) for $\delta < 0$ and $R_2 > \frac{1}{12}$, the new solvable potential of shape invariance in two steps. In the second solution, the superpotentials are shown to exhibit an $x^{-1}$ singularity with infinite discontinuity at the origin, together with a pair of antisymmetric finite discontinuities at some $\pm x \neq 0$ points. Hence, the corresponding regularized potential acquires three Dirac-delta function singularities, after proper regularization. We note that no solution can exist for the case of $\delta > 0$. The reason is that, from the general structure of energy spectrum, the system only allows a finite number of bound-states when $\delta \neq 0$. Unfortunately, for $\delta > 0$ it is infinitely high.

The construction of the potential algebra for the two superpotentials (29) and (30) is similar to that of subsection 3.1. We introduce the substitution of the parameter: $a_m \to \boldsymbol{\alpha}(N - m)$, where $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2\}$ and $N$ is an arbitrary integer. Then we use the identification (11) to write both superpotentials as $W(x, a_m) \equiv W\big(x, \boldsymbol{\alpha}\big(N - \frac{1}{2}(2m)\big)\big)$ and $\tilde{W}(x, a_m) \equiv W\big(x, \boldsymbol{\alpha}\big(N - \frac{1}{2}(2m + 1)\big)\big)$. In this way, both superpotentials share the same functional structure and are related to each other when the parameter $N$ of the unified superpotential is changed by $\frac{1}{2}$. Explicitly, the unified superpotential takes the compact form (for $s \equiv N - \frac{m}{2}$ and $m = 0, 1, 2, \ldots$)

$$W(x, \boldsymbol{\alpha}(s)) = \alpha_1(s)g(x) + \alpha_1\left(s + \frac{1}{2}\right)\tilde{g}(x) + \alpha_2(s)\frac{g^2(x)}{g(x) + \tilde{g}(x)}, \tag{32}$$

where two components of $\boldsymbol{\alpha}(s)$ are given by

$$\alpha_1(s) = \frac{1}{4R_2}(R_1 - (-1)^{2s}\tilde{R}_1) + \frac{1}{2}(1 + (-1)^{2s})[a_0 + (N - s)\delta], \tag{33}$$

$$\alpha_2(s) = -(-1)^{2s}\frac{\delta}{2}. \tag{34}$$

Using the definition of equation (32), we build the three angular-momentum-like generators of the associated potential algebra with ease

$$\begin{aligned} J_- &= \left[\frac{\partial}{\partial x} + \alpha_1(J_3)g(x) + \alpha_1\left(J_3 + \frac{1}{2}\right)\tilde{g}(x) + \alpha_2(J_3)\frac{g^2(x)}{g(x) + \tilde{g}(x)}\right]\mathrm{e}^{\mathrm{i}\phi/2}, \\ J_+ &= (J_-)^\dagger, \qquad J_3 = \mathrm{i}\frac{\partial}{\partial \phi}. \end{aligned} \tag{35}$$

It can be checked that the potential algebra based on the generators $J_-$, $J_+$ and $J_3$ is closed and nonlinear. In addition, we obtain the expression for the remainder

$$R(\boldsymbol{\alpha}(s)) = 4R_2\left\{(-1)^{2s}\mathcal{A}^2(s) + \left(1 - (-1)^{2s}\right)\left[\left(\frac{R_1 + \tilde{R}_1}{4R_2} + \frac{\delta}{4}\right)\mathcal{A}(s) - \frac{\delta^2}{32}\right]\right\}, \tag{36}$$

where the function $\mathcal{A}(s)$ is introduced to simplify the expression of $R(\boldsymbol{\alpha}(s))$. It is

$$\mathcal{A}(s) \equiv \tfrac{1}{4}\left[\alpha_1\left(s + \tfrac{1}{2}\right) + 2\alpha_1(s) + \alpha_1\left(s - \tfrac{1}{2}\right)\right]. \tag{37}$$

On solving equations (19) and (36), we deduce the coefficients $\varepsilon^2(\boldsymbol{\alpha}(s))$ of the form

$$\varepsilon^2(\boldsymbol{\alpha}(s)) = C - 2R_2\left\{(1 + (-1)^{2s})\mathcal{A}^2(s) + \frac{R_1 + \tilde{R}_1}{4\delta R_2}\left[\alpha_1(s) + \alpha_1\left(s - \frac{1}{2}\right)\right]^2\right\}, \tag{38}$$

where $C$ is a constant to render $\varepsilon^2(\boldsymbol{\alpha}(s))$ positive definite. Lastly, the eigenenergies of the initial Hamiltonian $H_-(a_0)$ can be constructed by equations (20) and (38).

## 4. Conclusions

In this paper, we explore the possible potential algebra of the shape invariance condition in two steps. By introducing the identification, the relations (11), connecting two different superpotentials of the shape invariant potentials in two steps, we obtain the simplified version of the associated potential algebra (16). This simplified potential algebra is similar to that of shape invariance in one step, and is based only on three angular-momentum-like generators, that is, $\{J_-, J_+, J_3\}$. The identification therefore reduces the shape invariance condition effectively from two steps to one step.

We also work out two classes of shape invariant potentials in two steps, in which the parameters $a_0$ and $a_1$ of partner potentials are related to each other by translation $a_1 = a_0 + \delta$. These shape invariant potentials are solvable but are singular, since discontinuity at some $x$-points is presented, giving rise to Dirac delta-function singularities. For each class of shape invariant potentials, the generators of the simplified potential algebra are explicitly constructed, which in turns provide an alternative way of determining energy eigenvalues by algebraic means.

Can the same technique be extended, for instance, to the shape invariance condition in more than two steps, therefore yielding the underlying simplified version of potential algebra? This does seem to be the case. We can directly generalize the identification (11) to $k$-step shape invariance condition. To see how it works, let us consider $k$ superpotentials, denoted by

$W_m(x, \alpha(N))$ with $m = 1, 2, \ldots, k$. In terms of these superpotentials, the shape invariance condition in $k$-steps is represented by the following $k$ relations:

$$\begin{aligned}
W_1^2(x, \alpha(N)) + W_1'(x, \alpha(N)) &= W_2^2(x, \alpha(N)) - W_2'(x, \alpha(N)) + R_1(\alpha(N)),\\
W_2^2(x, \alpha(N)) + W_2'(x, \alpha(N)) &= W_3^2(x, \alpha(N)) - W_3'(x, \alpha(N)) + R_2(\alpha(N)),\\
\cdots &= \cdots\\
W_k^2(x, \alpha(N)) + W_k'(x, \alpha(N)) &= W_1^2(x, \alpha(N-1)) - W_1'(x, \alpha(N-1)) + R_k(\alpha(N)).
\end{aligned}$$

These $k$ seemly unrelated relations can then be recast into a single constraint equation for the unified superpotential $W_1(x, \alpha(N))$, if we introduce the identification

$$W_m(x, \alpha(N)) \equiv W_1\left(x, \alpha\left(N - \frac{m-1}{k}\right)\right), \qquad R_m(\alpha(N)) \equiv R_1\left(\alpha\left(N - \frac{m-1}{k}\right)\right).$$

With the identification, we can easily reproduce the above $k$ relations among the $k$ superpotentials by projecting, one at a time, the operator equation

$$\begin{aligned}
&W_1^2(x, \alpha(J_3)) + W_1'(x, \alpha(J_3))\\
&\quad = W_1^2\left(x, \alpha\left(J_3 - \frac{1}{k}\right)\right) - W_1'\left(x, \alpha\left(J_3 - \frac{1}{k}\right)\right) + R_1(\alpha(J_3)),
\end{aligned}$$

into the $k$ different eigenstates $\left|N - \frac{m-1}{k}\right\rangle$, where $m = 1, 2, \ldots, k$. In this way, the shape invariance condition in $k$-steps is reduced to that in one step, and the simplified version of potential algebra is again based on three angular-momentum-like generators as

$$[J_3, J_\pm] = \pm\frac{1}{k} J_\pm, \qquad [J_+, J_-] = -R(\alpha(J_3))\,. \tag{39}$$

The representation of the associated potential algebra and the eigenenergies of the initial Hamiltonian can be constructed in a similar manner as those discussed in section 2. The example of solvable potentials of shape invariance in multi-steps that possess such simplified potential algebra is currently under investigation.

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